

# Subsystem Codes

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**Abstract**—We investigate various aspects of operator quantum error-correcting codes or, as we prefer to call them, subsystem codes. We give various methods to derive subsystem codes from classical codes. We give a proof for the existence of subsystem codes using a counting argument similar to the quantum Gilbert-Varshamov bound. We derive linear programming bounds and other upper bounds. We answer the question whether or not there exist  $[[n, n - 2d + 2, r > 0, d]]_q$  subsystem codes. Finally, we compare stabilizer and subsystem codes with respect to the required number of syndrome qudits.

## I. INTRODUCTION

Quantum error-correcting codes are seen as being indispensable for building a quantum computer. There are three predominant approaches to quantum error-correction: stabilizer codes, noiseless systems, and decoherence free subspaces. Recent advances in the theory of quantum error-correction have shown that all these apparently disparate approaches are actually the same. This unification goes under the name of operator quantum error-correction codes [1], [7]–[11], though we will prefer to use the shorter and more descriptive term *subsystem codes*. Subsystem codes provide a common platform for comparing the various different types of quantum codes and make it possible to treat active and passive quantum error-correction within the same framework. Apart from the fact that subsystem codes give us more control over the degree of passive error-correction, there have been claims that subsystem codes can make quantum error-correction more robust and practical. For example, it has been claimed that subsystem codes make it possible to derive simpler error recovery schemes in comparison to stabilizer codes. Furthermore, it is conjectured that certain subsystem codes are self-correcting [1].

Subsystem codes are relatively new and promise to be a fruitful area for quantum error-correction. Until now, there are few concrete examples of such codes and even fewer systematic code constructions. Little is known about the parameters of subsystem codes, so it is difficult to judge the performance of such codes.

In a recent work [6], we derived a character-theoretic framework for the construction of subsystem codes. We were able to show that there exists a correspondence between the subsystem codes over  $\mathbb{F}_q$  and classical additive codes over  $\mathbb{F}_q$  and  $\mathbb{F}_{q^2}$ . In this paper, we investigate basic properties of subsystem codes, establish further connections to classical codes, and derive bounds on the parameters of subsystem

codes. We report first results on a fair comparison between stabilizer codes and subsystem codes.

The paper is structured as follows. After a brief introduction to subsystem codes in Section II, we recall some results about the relations between subsystem codes and classical codes. Then we give some simple constructions of subsystem codes which parallel the common constructions of stabilizer codes. In Section III, we give a nonconstructive proof of the existence of subsystem codes. In Section IV, we derive linear programming upper bounds on the parameters of subsystem codes. For pure subsystem codes (to be defined later) we can also derive analytical upper bounds which resemble the quantum Singleton and Hamming bounds. Armed with these results on bounds we make a rigorous comparison of stabilizer codes and subsystem codes, that makes precise when subsystem codes can do better.

*Notation:* We assume that  $q$  is the power of a prime  $p$  and  $\mathbb{F}_q$  denotes a finite field with  $q$  elements. By qudit we mean a  $q$ -ary quantum bit. The symplectic weight of an element  $w = (x_1, \dots, x_n, y_1, \dots, y_n)$  in  $\mathbb{F}_q^{2n}$  is defined as  $\text{swt}(w) = |\{(x_i, y_i) \neq (0, 0) \mid 1 \leq i \leq n\}|$ . The trace-symplectic product of two elements  $u = (a|b), v = (a'|b')$  in  $\mathbb{F}_q^{2n}$  is defined as  $\langle u|v \rangle_s = \text{tr}_{q/p}(a' \cdot b - a \cdot b')$ , where  $x \cdot y$  is the usual euclidean inner product. The trace-symplectic dual of a code  $C \subseteq \mathbb{F}_q^{2n}$  is defined as  $C^\perp_s = \{v \in \mathbb{F}_q^{2n} \mid \langle v|w \rangle_s = 0 \text{ for all } w \in C\}$ . For vectors  $x, y$  in  $\mathbb{F}_q^n$ , we define the hermitian inner product  $\langle x|y \rangle_h = \sum_{i=1}^n x_i^q y_i$  and the hermitian dual of  $C \subseteq \mathbb{F}_q^n$  as  $C^\perp_h = \{x \in \mathbb{F}_q^n \mid \langle x|y \rangle_h = 0 \text{ for all } y \in C\}$ . The trace alternating form of two vectors  $u, w$  in  $\mathbb{F}_q^n$  is defined as  $\langle u|v \rangle_a = \text{tr}_{q/p}[(\langle u|v \rangle_h - \langle v|u \rangle_h)/(\beta^2 - \beta^{2q})]$ , where  $\{\beta, \beta^q\}$  is a normal basis of  $\mathbb{F}_{q^2}$  over  $\mathbb{F}_q$ . If  $C \subseteq \mathbb{F}_{q^2}^n$ , then the trace alternating dual of  $C$  is defined as  $C^\perp_a = \{x \in \mathbb{F}_{q^2}^n \mid \langle x|y \rangle_a = 0 \text{ for all } y \in C\}$ .

## II. SUBSYSTEM CODES AND CLASSICAL CODES

Let  $\mathcal{H} = \mathbb{C}^q \otimes \mathbb{C}^q \otimes \dots \otimes \mathbb{C}^q = \mathbb{C}^{q^n}$ . An orthonormal basis for  $\mathbb{C}^{q^n}$  is  $B = \{|x\rangle \mid x \in \mathbb{F}_q^n\}$ . The vector  $|x\rangle = |x_1\rangle \otimes |x_2\rangle \otimes \dots \otimes |x_n\rangle$ . The elements of  $\mathcal{H}$  are of the form

$$v = \sum_{x \in \mathbb{F}_q^n} v_x |x\rangle \text{ where } v_x \in \mathbb{C} \text{ and } \sum_{x \in \mathbb{F}_q^n} |v_x|^2 = 1.$$

We define the following unitary operators on  $\mathbb{C}^q$

$$X_a |x\rangle = |x + a\rangle \text{ and } Z_b |x\rangle = \omega^{\text{tr}_{q/p}(bx)} |x\rangle,$$

where  $\omega = e^{j2\pi/p}$ . The set of errors  $\mathcal{E} = \{X_a Z_b \mid a, b \in \mathbb{F}_q\}$  form a basis for errors on a single qudit. Every error on a single qudit can be expressed as linear combination of the elements in  $\mathcal{E}$ . If we assume that the errors are independent

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on each qudit, we need only consider the error group  $E = \{\omega^c e_1 \otimes e_2 \otimes \cdots \otimes e_n \mid c \in \mathbb{F}_p, e_i \in \mathcal{E}\}$ , where each of the  $e_i$  is a single qudit error. The weight of an error is the number of qudits that are in error. For further details on the error model and the actual structure of the error group we refer the reader to [5].

A quantum error-correcting code  $Q$  is a subspace in  $\mathcal{H} = \mathbb{C}^{q^n}$  such that  $\mathcal{H} = Q \oplus Q^\perp$ , where  $Q^\perp$  is the orthogonal complement of  $Q$ . In a subsystem code, the subspace  $Q$  further decomposes into a tensor product of two vector space  $A$  and  $B$ , that is,

$$Q = A \otimes B.$$

The vector spaces  $A$  and  $B$  are respectively called the subsystem and the co-subsystem of the code  $Q$ . The information to be protected is stored in the subsystem  $A$ , whence the name subsystem code.

If  $\dim A = K$ ,  $\dim B = R$  and  $Q$  is able to detect all errors in  $E$  of weight less than  $d$  on subsystem  $A$ , then we say that  $Q$  is an  $((n, K, R, d))_q$  subsystem code. We call  $d$  the minimum distance of the subsystem  $A$  or, by slight abuse of language, the minimum distance of the subsystem code  $Q$  (when the tensor decomposition  $Q = A \otimes B$  is understood from the context). We write  $[[n, k, r, d]]_q$  for an  $((n, q^k, q^r, d))_q$  subsystem code. Sometimes we will say that an  $[[n, k, r, d]]_q$  subsystem code has  $r$  virtual gauge qudits, which is simply another way of saying that the dimension of the co-subsystem is  $q^r$ ; it should be stressed that the gauge qudits typically do not correspond to physical qudits.

#### A. Subsystem Codes From Classical Codes

We recall the following results from [6] which relate quantum subsystem codes to classical codes.

**Theorem 1:** Let  $X$  be a classical additive subcode of  $\mathbb{F}_q^{2n}$  such that  $X \neq \{0\}$  and let  $Y$  denote its subcode  $Y = X \cap X^{\perp_s}$ . If  $x = |X|$  and  $y = |Y|$ , then there exists subsystem code  $C = A \otimes B$  such that

- i)  $\dim A = q^n/(xy)^{1/2}$ ,
- ii)  $\dim B = (x/y)^{1/2}$ .

The minimum distance of subsystem  $A$  is given by  $d = \text{wt}((X + X^{\perp_s}) - X) = \text{wt}(Y^{\perp_s} - X)$ . Thus, the subsystem  $A$  can detect all errors in  $E$  of weight less than  $d$ , and can correct all errors in  $E$  of weight  $\leq \lfloor (d-1)/2 \rfloor$ .

*Proof:* See [6, Theorem 5]. ■

**Remark 1:** Recall that  $|X^{\perp_s}| = q^{2n}/|X|$ . Therefore, the dimension of the subsystem  $A$  can also be calculated as  $\dim A = (|X^{\perp_s}|/|Y|)^{1/2}$ .

It is also possible to construct subsystem codes via codes over  $\mathbb{F}_{q^2}$  using  $\langle \cdot | \cdot \rangle_a$ , the trace alternating form [5] which gives us the following theorem. The proof can be found in [6, Theorem 6].

**Theorem 2:** Let  $X$  be a classical additive subcode of  $\mathbb{F}_{q^2}^n$  such that  $X \neq \{0\}$  and let  $Y$  denote its subcode  $Y = X \cap X^{\perp_a}$ . If  $x = |X|$  and  $y = |Y|$ , then there exists subsystem code  $C = A \otimes B$  such that

- i)  $\dim A = q^n/(xy)^{1/2}$ ,
- ii)  $\dim B = (x/y)^{1/2}$ .

The minimum distance of subsystem  $A$  is given by

$$d = \text{wt}((X + X^{\perp_a}) - X) = \text{wt}(Y^{\perp_a} - X),$$

where  $\text{wt}$  denotes the Hamming weight. Thus, the subsystem  $A$  can detect all errors in  $E$  of Hamming weight less than  $d$ , and can correct all errors in  $E$  of Hamming weight  $\lfloor (d-1)/2 \rfloor$  or less.

*Proof:* This follows from Theorem 1 and the fact that there exists a weight-preserving isometric isomorphism from  $(\mathbb{F}_q^{2n}, \langle \cdot | \cdot \rangle_s)$  and  $(\mathbb{F}_{q^2}^n, \langle \cdot | \cdot \rangle_a)$ , see [5]. ■

Theorem 2 has the advantage that the weights of the codes over  $\mathbb{F}_{q^2}$  is measured using the usual Hamming distance.

We are now going to derive some particularly important special cases of the above two theorems as a consequence. Before stating these results, we recall the following simple fact.

**Lemma 3:** Let  $C_1$  and  $C_2$  be two  $\mathbb{F}_q$ -linear codes of length  $n$ . The product code  $C_1 \times C_2 = \{(a|b) \mid a \in C_1, b \in C_2\}$  has length  $2n$  and its trace-symplectic dual is given by

$$(C_1 \times C_2)^{\perp_s} = C_2^\perp \times C_1^\perp.$$

*Proof:* If  $(a|b) \in C_1 \times C_2$  and  $(a'|b') \in C_2^\perp \times C_1^\perp$ , then  $\text{tr}_{q/p}(b \cdot a' - b' \cdot a) = 0$ ; hence,  $C_2^\perp \times C_1^\perp \subseteq (C_1 \times C_2)^{\perp_s}$ . Comparing dimensions shows that equality must hold. ■

The first consequence uses the euclidean inner product, that is, the usual dot inner product on  $\mathbb{F}_q^n$  to construct subsystem codes. In the special case of stabilizer codes, this yields the well-known CSS construction (for instance, see [3, Theorem 9]).

**Corollary 4 (Euclidean Construction):** Let  $C_i \subseteq \mathbb{F}_q^n$ , be  $[[n, k_i]]_q$  linear codes where  $i \in \{1, 2\}$ . Then there exists an  $[[n, k, r, d]]_q$  subsystem code with

- $k = n - (k_1 + k_2 + k')/2$ ,
  - $r = (k_1 + k_2 - k')/2$ , and
  - $d = \min\{\text{wt}((C_1^\perp \cap C_2)^\perp \setminus C_1), \text{wt}((C_2^\perp \cap C_1)^\perp \setminus C_2)\}$ ,
- where  $k' = \dim_{\mathbb{F}_q}(C_1 \cap C_2^\perp) \times (C_1^\perp \cap C_2)$ .

*Proof:* Let  $C = C_1 \times C_2$ , then by Lemma 3,  $C^{\perp_s} = C_2^\perp \times C_1^\perp$ , and  $D = C \cap C^{\perp_s} = (C_1 \cap C_2^\perp) \times (C_2 \cap C_1^\perp)$ . Again by Lemma 3,  $D^{\perp_s} = (C_2 \cap C_1^\perp)^\perp \times (C_1 \cap C_2^\perp)^\perp$ . Let  $\dim_{\mathbb{F}_q} D = k'$ . Then  $|C||D| = q^{k_1+k_2+k'}$  and  $|C|/|D| = q^{k_1+k_2-k'}$ . By Theorem 1, the code  $C$  defines an  $[[n, n - (k_1 + k_2 + k')/2, (k_1 + k_2 - k')/2, d]]_q$  subsystem code. The distance of the code is given by

$$\begin{aligned} d &= \text{wt}(D^{\perp_s} \setminus C) \\ &= \text{wt}((C_2 \cap C_1^\perp)^\perp \times (C_1 \cap C_2^\perp)^\perp \setminus (C_1 \times C_2)). \end{aligned}$$

The latter expression can be simplified to

$$d = \min\{\text{wt}((C_2 \cap C_1^\perp)^\perp \setminus C_1), \text{wt}((C_1 \cap C_2^\perp)^\perp \setminus C_2)\},$$

which proves the claim. ■

Setting  $C_2 = C_1$  in the previous construction simplifies the computation of the code parameters. Then we have an  $[[n, n - k - k', k - k', \text{wt}((C_1 \cap C_1^\perp)^\perp \setminus C_1)]]_q$  code, where  $k' = \dim_{\mathbb{F}_q} C_1 \cap C_1^\perp$ . Therefore, any family of classical codes where the dimension of  $C_1 \cap C_1^\perp$  and the minimum distance of the dual of  $C_1 \cap C_1^\perp$  is known, will provide us with a family of subsystem codes. The codes that arise when  $C_1 =$

$C_2$  will also arise as a special case of the next construction.

**Corollary 5 (Hermitian Construction):** Let  $C \subseteq \mathbb{F}_{q^2}^n$  be an  $\mathbb{F}_{q^2}$ -linear  $[n, k, d]_{q^2}$  code such that  $D = C \cap C^{\perp_h}$  is of dimension  $k' = \dim_{\mathbb{F}_{q^2}} D$ . Then there exists an

$$[[n, n - k - k', k - k', \text{wt}(D^{\perp_h} \setminus C)]]_q$$

subsystem code.

*Proof:* If  $C$  is linear, then  $C^{\perp_a} = C^{\perp_h}$  by [6, Lemma 18]. It follows that  $q^n / \sqrt{|D||C|} = q^{n-k'-k}$  and  $\sqrt{|C|/|D|} = q^{k-k'}$ . Let  $d = \text{wt}(D^{\perp_h} \setminus C)$ . Then, by Theorem 2, there exists an  $[[n, n - k - k', k - k', d]]_q$  subsystem code. ■

The subsystem codes can be easily constructed with the help of a computer algebra system. The following example gives some subsystem codes constructed using MAGMA [2].

**Example 1 (BCH Subsystem Codes):** The binary subsystem codes in Table I were derived from BCH codes over  $\mathbb{F}_4$  via Corollary 5.

TABLE I  
BCH SUBSYSTEM CODES

Subsystem Code	Parent BCH Code	Designed distance
$[[15, 1, 2, 5]]_2$	$[15, 8, 6]_{2^2}$	6
$[[15, 5, 2, 3]]_2$	$[15, 6, 7]_{2^2}$	7
$[[17, 8, 1, 4]]_2$	$[17, 5, 9]_{2^2}$	4
$[[21, 6, 3, 3]]_2$	$[21, 9, 7]_{2^2}$	6
$[[21, 7, 2, 3]]_2$	$[21, 8, 9]_{2^2}$	8
$[[31, 10, 1, 5]]_2$	$[31, 11, 11]_{2^2}$	8
$[[31, 20, 1, 3]]_2$	$[31, 6, 15]_{2^2}$	12

Codes constructed with the help of Corollaries 4 and 5 will lead to  $\mathbb{F}_q$ -linear and  $\mathbb{F}_{q^2}$ -linear subsystem codes respectively. Though in some cases Corollary 4 can lead to  $\mathbb{F}_{q^2}$ -linear codes. So when we refer to a subsystem code as being  $\mathbb{F}_q$ -linear, it could be also  $\mathbb{F}_{q^2}$ -linear. In this paper, we will call a subsystem code that can be constructed with the help of Theorems 1 and 2 and their corollaries, a *Clifford subsystem code*. Ten years ago, Knill suggested a generalization of stabilizer codes that became known as Clifford codes (because their construction uses a part of representation theory known as Clifford theory). Recently, we realized that a special case of Knill's construction leads to a very natural construction of subsystem codes. Clifford theory is the natural tool in the construction of these subsystem codes, whence the name.

### III. LOWER BOUNDS ON SUBSYSTEM CODES

In this section we give a simple nonconstructive proof for the existence of subsystem codes. The proof is based on a counting argument similar to the quantum Gilbert-Varshamov bound for stabilizer codes [5]. We will need the following simple fact.

**Lemma 6:** Let  $\mathbb{F}_q$  be a finite field of characteristic  $p$ . Let  $r$  and  $s$  be nonnegative integers such that  $p^{r+2s} \leq q^{2n}$ . Then there exists an additive subcode  $X$  of  $\mathbb{F}_q^{2n}$  such that  $|X| = p^{r+2s}$  and  $|X \cap X^{\perp_s}| = p^r$ .

*Proof:* Let  $m$  denote the integer such that  $q = p^m$ . We may regard  $\mathbb{F}_q^{2n}$  as an  $2nm$ -dimensional vector space over  $\mathbb{F}_p$ . Then  $\langle \cdot | \cdot \rangle_s$  is a nondegenerate skew-symmetric bilinear form on this vector space. Therefore, there exists a direct sum decomposition of  $\mathbb{F}_q^{2n} \cong \mathbb{F}_p^{2nm} = V_1 \oplus \cdots \oplus V_{nm}$ , where  $V_k$  is a 2-dimensional subspace with basis  $\{x_k, z_k\}$  such that  $\langle x_k | x_\ell \rangle_s = 0 = \langle z_k | z_\ell \rangle_s$  for  $1 \leq k, \ell \leq nm$ ,  $\langle x_k | z_k \rangle_s \neq 0$ , and  $\langle x_k | z_\ell \rangle_s = 0$  if  $k \neq \ell$ . Then  $X = \langle z_1, \dots, z_r, x_{r+1}, z_{r+1}, \dots, x_{r+s}, z_{r+s} \rangle$  is a code with the desired properties. ■

**Theorem 7:** Let  $\mathbb{F}_q$  be a finite field of characteristic  $p$ . If  $K$  and  $R$  are powers of  $p$  such that  $1 < KR \leq q^n$  and  $d$  is a positive integer such that

$$\sum_{j=1}^{d-1} \binom{n}{j} (q^2 - 1)^j (q^n KR - q^n R/K) < (p-1)(q^{2n} - 1)$$

holds, then an  $((n, K, R, \geq d))_q$  subsystem code exists.

*Proof:* By Lemma 6, there exists an additive subcode  $X$  of  $\mathbb{F}_q^{2n}$  such that  $x = |X| = q^n R/K$  and  $y = |X \cap X^{\perp_s}| = q^n/(KR)$ ; the resulting subsystem code has a subsystem of dimension  $q^n/(xy)^{1/2} = K$  and a co-subsystem of dimension  $(x/y)^{1/2} = R$ . Therefore, the multiset  $\mathcal{X}$  given by

$$\mathcal{X} = \left\{ (X + X^{\perp_s}) - X \mid \begin{array}{l} X \text{ is an additive subcode of} \\ \mathbb{F}_q^{2n} \text{ such that } |X| = q^n R/K \\ \text{and } |X \cap X^{\perp_s}| = q^n/(KR) \end{array} \right\}$$

is not empty.

Thus, an element of  $\mathcal{X}$  corresponds to a subsystem code  $C = A \otimes B$  with  $\dim A = K$  and  $\dim B = R$ . The set difference  $(X + X^{\perp_s}) - X$  contains only nonzero vectors of  $\mathbb{F}_q^{2n}$ . We claim that all nonzero vector in  $\mathbb{F}_q^{2n}$  appear in the same number of sets in  $\mathcal{X}$ . Indeed, the symplectic group  $\text{Sp}(2n, \mathbb{F}_q)$  acts transitively on the set  $\mathbb{F}_q^{2n} \setminus \{0\}$ , see [4, Proposition 3.2], which means that for any nonzero vectors  $u$  and  $v$  in  $\mathbb{F}_q^{2n}$  there exists  $\tau \in \text{Sp}(2n, \mathbb{F}_q)$  such that  $v = \tau u$ . Therefore,  $u$  is contained in  $(X + X^{\perp_s}) - X$  if and only if  $v$  is contained in the element  $(\tau X + (\tau X)^{\perp_s}) - \tau X$  of  $\mathcal{X}$ .

Since  $|(X + X^{\perp_s}) - X| = q^n KR - q^n R/K$ , we can conclude that any nonzero vector of  $\mathbb{F}_q^{2n}$  occurs in  $|\mathcal{X}|(q^n KR - q^n R/K)/(q^{2n} - 1)$  elements of  $\mathcal{X}$ . Furthermore, a nonzero vector and its  $\mathbb{F}_p^\times$ -multiples are contained in the exact same sets of  $\mathcal{X}$ . Therefore, if we delete all sets from  $\mathcal{X}$  that contain a nonzero vector with symplectic weight less than  $d$ , then we remove at most

$$\frac{\sum_{j=1}^{d-1} \binom{n}{j} (q^2 - 1)^j}{p-1} |\mathcal{X}| \frac{(q^n KR - q^n R/K)}{q^{2n} - 1}$$

sets from  $\mathcal{X}$ . By assumption, this number is less than  $|\mathcal{X}|$ ; hence, there exists an  $((n, K, R, \geq d))_q$  subsystem code. ■ The lower bound has important implications for comparing stabilizer codes with subsystem codes as we shall see in Section VI. Further, we obtain the following lower bound for stabilizer codes as a simple corollary, when  $R = 1$  (see also [5]).

*Corollary 8 (GV Bound for Stabilizer Codes):* Let  $\mathbb{F}_q$  be a finite field of characteristic  $p$  and  $1 < K \leq q^n$  a power of  $p$ . If

$$\sum_{j=1}^{d-1} \binom{n}{j} (q^2 - 1)^j < (p - 1) \frac{(q^{2n} - 1)}{(q^n K - q^n / K)}$$

holds, then an  $((n, K, \geq d))_q$  stabilizer code exists.

A stronger result showing the existing of linear stabilizer codes was shown in [5, Lemma 31].

#### IV. UPPER BOUNDS FOR SUBSYSTEM CODES

We want to investigate some limitations on subsystem codes that can be constructed with the help of Theorem 1 (or, equivalently, Theorem 2). To that end, we will investigate some upper bounds on the parameters of subsystem codes.

##### A. Linear Programming Bounds

*Theorem 9:* If an  $((n, K, R, d))_q$  Clifford subsystem code with  $K > 1$  exists, then there exists a solution to the optimization problem: maximize  $\sum_{j=1}^{d-1} A_j$  subject to the constraints

- 1)  $A_0 = B_0 = 1$  and  $0 \leq B_j \leq A_j$  for all  $1 \leq j \leq n$ ;
- 2)  $\sum_{j=0}^n A_j = q^n R / K$ ;  $\sum_{j=0}^n B_j = q^n / K R$ ;
- 3)  $A_j^{\perp s} = \frac{K}{q^n R} \sum_{r=0}^n K_j(r) A_r$  holds for all  $j$  in the range  $0 \leq j \leq n$ ;
- 4)  $B_j^{\perp s} = \frac{K R}{q^n} \sum_{r=0}^n K_j(r) B_r$  holds for all  $j$  in the range  $0 \leq j \leq n$ ;
- 5)  $A_j = B_j^{\perp s}$  for all  $j$  in  $0 \leq j < d$  and  $A_j \leq B_j^{\perp s}$  for all  $d \leq j \leq n$ ;
- 6)  $B_j = A_j^{\perp s}$  for all  $j$  in  $0 \leq j < d$  and  $B_j \leq A_j^{\perp s}$  for all  $d \leq j \leq n$ ;
- 7)  $(p - 1)$  divides  $A_j, B_j, A_j^{\perp s}$ , and  $B_j^{\perp s}$  for all  $j$  in the range  $1 \leq j \leq n$ ;

where the coefficients  $A_j$  and  $B_j$  assume only integer values, and  $K_j(r)$  denotes the Krawtchouk polynomial

$$K_j(r) = \sum_{s=0}^j (-1)^s (q^2 - 1)^{j-s} \binom{r}{s} \binom{n-r}{j-s}.$$

*Proof:* If an  $((n, K, R, d))_q$  subsystem code exists, then the weight distribution  $A_j$  of the associated additive code  $X$  and the weight distribution  $B_j$  of its subcode  $Y = X \cap X^{\perp s}$  obviously satisfy 1). By Theorem 1, we have  $K = q^n / \sqrt{|X||Y|}$  and  $R = \sqrt{|X|/|Y|}$ , which implies  $|X| = \sum A_j = q^n R / K$  and  $|Y| = \sum B_j = q^n / K R$ , proving 2). Conditions 3) and 4) follow from the MacWilliams relation for symplectic weight distribution, see [5, Theorem 23]. As  $X$  is an  $\mathbb{F}_p$ -linear code, for each nonzero codeword  $c$  in  $X$ ,  $\alpha c$  is again in  $X$  for all  $\alpha$  in  $\mathbb{F}_p^\times$ ; thus, condition 7) must hold. Since the quantum code has minimum distance  $d$ , all vectors of symplectic weight less than  $d$  in  $Y^{\perp s}$  must be in  $X$ , since  $Y^{\perp s} - X$  has minimum distance  $d$ ; this implies 5). Similarly, all vectors in  $X^{\perp s} \subseteq X + X^{\perp s}$  of symplectic weight less

than  $d$  must be contained in  $X$ , since  $(X + X^{\perp s}) - X$  has minimum distance  $d$ ; this implies 6). ■

We can use the previous theorem to derive bounds on the dimension of the co-subsystem. If the optimization problem is not solvable, then we can immediately conclude that a code with the corresponding parameter settings cannot exist.

Perhaps one of the most striking features of subsystem codes is the potential reduction of syndrome measurements. Recall that an  $\mathbb{F}_q$ -linear  $[[n, k, d]]_q$  stabilizer code requires  $n - k$  syndrome measurements. On the other hand, an  $\mathbb{F}_q$ -linear  $[[n, k, r, d]]_q$  Clifford subsystem code requires just  $n - k - r$  syndrome measurements.

Poulin [11] asked whether we can have  $[[5, 1, r > 0, 3]]_2$  Clifford subsystem code. Of course, such a code would be preferable over the  $[[5, 1, 3]]_2$  stabilizer code. After an exhaustive computer search, he concluded that such a subsystem code does not exist. This result can be obtained very easily with the linear programming bounds. In fact, our investigations for small lengths revealed that not only a  $[[5, 1, r > 0, 3]]_2$  code does not exist, but neither does any code with parameters given in the next example.

*Example 2:* Theorem 9 shows that it is not possible to construct subsystem codes with  $r > 0$  and parameters shown in Table II.

TABLE II

Field	Codes
$\mathbb{F}_2$	$[[4, 2, r, 2]]_2, [[5, 1, r, 3]]_2$
$\mathbb{F}_3$	$[[4, 2, r, 2]]_3, [[5, 1, r, 3]]_3,$ $[[9, 3, r, 4]]_3, [[9, 5, r, 3]]_3,$ $[[10, 6, r, 3]]_3$
$\mathbb{F}_4$	$[[4, 2, r, 2]]_4, [[5, 1, r, 3]]_4,$ $[[9, 3, r, 4]]_4, [[9, 5, r, 3]]_4,$ $[[10, 6, r, 3]]_4$

The previous example is motivated by the fact that one can improve upon Shor's  $[[9, 1, 3]]_2$  quantum stabilizer code by allowing three additional gauge qubits, that is, there exists a  $[[9, 1, 3, 3]]_2$  subsystem code, see [11]. The practical relevance is that the  $9 - 1 = 8$  syndrome measurements that are required for Shor's code are reduced to  $9 - 1 - 3 = 5$  syndrome measurements in the subsystem code.

Since we allow nonbinary alphabets in this paper, a natural generalization of Poulin's question is whether one can find an  $[[n, n - 2d + 2, r, d]]_q$  subsystem code with  $r > 0$ . The above example shows that such subsystem codes with such parameters do not exist for certain small lengths and small alphabet sizes.

We will fully answer this question in the subsequent sections. In the search for an answer to this problem, we were prompted to define the notion of pure subsystem codes. The notion of purity proved to be fruitful in deducing this and other results.

##### B. Pure Subsystem Codes

Let  $X$  be an additive subcode of  $\mathbb{F}_q^{2n}$  and  $Y = X \cap X^{\perp s}$ . By Theorem 1, we can obtain an  $((n, K, R, d))_q$  subsystem



code  $Q$  from  $X$  that has minimum distance  $d = \text{swt}(Y^{\perp_s} - X)$ . The set difference involved in the definition of the minimum distance makes it harder to compute the minimum distance. Therefore, we introduce pure codes that are easier to analyze.

We say that the subsystem code  $Q$  is *pure to  $d'$*  if  $d' \leq \text{swt}(X)$ . The code is *exactly pure to  $d'$*  if it is pure to  $d'$  but not to  $d' + 1$ ; then  $\text{swt}(X) = d'$ . Any subsystem code is always exactly pure to  $d' = \text{swt}(X)$ . We call  $Q$  a pure subsystem code if it is pure to  $d' \geq d$ ; otherwise, we call  $Q$  an impure subsystem code. Pure codes do not require us to compute the minimum distance of the difference set  $Y^{\perp_s} - X$ . We can compute the distance of the code as  $d = \text{swt}(Y^{\perp_s})$ , which is comparatively simpler task though it is also computationally hard.

The purity of codes over  $\mathbb{F}_{q^2}$  is defined in a similar way.

*Example 3 (Reed-Solomon Subsystem Codes):* The non-binary subsystem codes given in Table III are all pure and were derived from primitive narrow-sense Reed-Solomon codes over  $\mathbb{F}_{q^2}$ . It is curious that the distance of many of

TABLE III  
REED-SOLOMON SUBSYSTEM CODES

Subsystem Codes	Parent RS Code
$[[15, 1, 10, 3]]_4$	$[15, 12, 4]_{4^2}$
$[[15, 1, 8, 3]]_4$	$[15, 11, 5]_{4^2}$
$[[15, 1, 6, 3]]_4$	$[15, 10, 6]_{4^2}$
$[[15, 2, 5, 3]]_4$	$[15, 9, 7]_{4^2}$
$[[24, 1, 17, 4]]_5$	$[24, 20, 5]_{5^2}$
$[[24, 2, 10, 4]]_5$	$[24, 16, 9]_{5^2}$
$[[24, 4, 10, 4]]_5$	$[24, 15, 10]_{5^2}$
$[[24, 16, 2, 4]]_5$	$[24, 5, 20]_{5^2}$
$[[24, 17, 1, 4]]_5$	$[24, 4, 21]_{5^2}$
$[[24, 19, 1, 3]]_5$	$[24, 3, 22]_{5^2}$
$[[48, 1, 37, 6]]_7$	$[48, 42, 7]_{7^2}$
$[[48, 2, 26, 6]]_7$	$[48, 36, 13]_{7^2}$

these subsystem codes is equal to  $q - 1$ . We conjecture that, in general, the distance of a subsystem code constructed from a Reed-Solomon code over  $\mathbb{F}_{q^2}^{q^2-1}$  cannot exceed  $q - 1$ .

### C. Upper Bounds for Pure Subsystem Codes

In this subsection, we establish a number of basic results concerning pure subsystem codes. The next lemma is a key result that associates to a pure subsystem code a pure stabilizer code.

*Lemma 10:* If a pure  $((n, K, R, d))_q$  Clifford subsystem code  $Q$  exists, then there exists a pure  $((n, KR, d))_q$  stabilizer code.

*Proof:* Let  $X$  be a classical additive subcode of  $\mathbb{F}_q^{2n}$  that defines  $Q$ , and let  $Y = X \cap X^{\perp_s}$ . Furthermore, Theorem 1 implies that  $KR = q^n/|Y|$ . Since  $Y \subseteq Y^{\perp_s}$ , there exists an  $((n, q^n/|Y|, d'))_q$  stabilizer code with minimum distance  $d' = \text{wt}(Y^{\perp_s} - Y)$ . The purity of  $Q$  implies that  $\text{swt}(Y^{\perp_s} - X) = \text{swt}(Y^{\perp_s}) = d$ . As  $Y \subseteq X$ , it follows that  $d' = \text{swt}(Y^{\perp_s} - Y) = \text{swt}(Y^{\perp_s}) = d$ ; hence, there exists a pure  $((n, KR, d))_q$  stabilizer code. ■

As a consequence of the preceding lemma, it is straightforward to obtain the following bounds on pure subsystem codes.

*Theorem 11:* Any pure  $((n, K, R, d))_q$  Clifford subsystem code satisfies  $KR \leq q^{n-2d+2}$ .

*Proof:* By Lemma 10, there exists a pure  $((n, KR, d))_q$  stabilizer code. By the quantum Singleton bound, we have  $KR \leq q^{n-2d+2}$ . ■

*Corollary 12:* A pure  $[[n, k, r, d]]_q$  Clifford subsystem code satisfies  $k + r \leq n - 2d + 2$ .

*Example 4 (Optimal Subsystem Codes):* All the following codes constructed from Reed-Solomon codes over  $\mathbb{F}_{q^2}$  are pure and meet the bound in Theorem 11. These codes are in that sense optimal subsystem codes.

TABLE IV  
OPTIMAL PURE SUBSYSTEM CODES

Subsystem Codes	Parent Code (RS Code)
$[[15, 1, 10, 3]]_4$	$[15, 12, 4]_{4^2}$
$[[15, 9, 2, 3]]_4$	$[15, 4, 12]_{4^2}$
$[[15, 10, 1, 3]]_4$	$[15, 3, 13]_{4^2}$
$[[24, 1, 17, 4]]_5$	$[24, 20, 5]_{5^2}$
$[[24, 16, 2, 4]]_5$	$[24, 5, 20]_{5^2}$
$[[24, 17, 1, 4]]_5$	$[24, 4, 21]_{5^2}$
$[[24, 19, 1, 3]]_5$	$[24, 3, 22]_{5^2}$
$[[48, 1, 37, 6]]_7$	$[48, 42, 7]_{7^2}$

We can also show that the pure subsystem codes obey a quantum Hamming bound like the stabilizer codes. We skip the proof as it is along the same lines as Theorem 11.

*Lemma 13:* A pure  $((n, K, R, d))_q$  Clifford subsystem code satisfies

$$\sum_{j=0}^{\lfloor \frac{d-1}{2} \rfloor} \binom{n}{j} (q^2 - 1)^j \leq q^n / KR.$$

## V. SUBSYSTEM CODE CONSTRUCTIONS

In this section, we give new constructions for pure subsystem codes. We begin with a proof of the simple, yet surprising, observation that one can always exchange information qudits and gauge qudits in the case of pure subsystem codes.

*Lemma 14:* If there exists a pure  $((n, K, R, d))_q$  Clifford subsystem code, then there also exists an  $((n, R, K, \geq d))_q$  Clifford subsystem code that is pure to  $d$ .

*Proof:* By Theorem 2, there exist classical codes  $D \subseteq C \subseteq \mathbb{F}_{q^2}^n$  with the parameters  $(n, q^n R / K)_{q^2}$  and  $(n, q^n / KR)_{q^2}$ . Furthermore, since the subsystem code is pure, we have  $\text{wt}(D^{\perp_a} \setminus C) = \text{wt}(D^{\perp_a}) = d$ .

Let us interchange the roles of  $C$  and  $C^{\perp_a}$ , that is, now we construct a subsystem code from  $C^{\perp_a}$ . The parameters of the resulting subsystem code are given by

$$((n, \sqrt{|D^{\perp_a}|/|C^{\perp_a}|}, \sqrt{|C^{\perp_a}|/|D|}, \text{wt}(D^{\perp_a} \setminus C^{\perp_a})))_q.$$

We note that

- $\sqrt{|D^{\perp_a}|/|C^{\perp_a}|} = \sqrt{|C|/|D|} = R$  and
- $\sqrt{|C^{\perp_a}|/|D|} = \sqrt{|D^{\perp_a}|/|C|} = K$ .

The minimum distance  $d'$  of the resulting code satisfies  $d' = \text{wt}(D^{\perp_a} \setminus C^{\perp_a}) \geq \text{wt}(D^{\perp_a}) = d$ ; the claim about the purity follows from the fact that  $\text{wt}(D^{\perp_a}) = d$ . ■

Before proving our next result, we need the following fact from linear algebra.

*Lemma 15:* Let  $\mathbb{F}_q$  be a finite field of characteristic  $p$ . Let  $C$  denote an additive subcode of  $\mathbb{F}_q^{2n}$ . There exists an  $\mathbb{F}_p$ -basis  $B$  generating the code  $C$  that is of the form

$$B = \{z_1, x_1; \dots; z_r, x_r; z_{r+1}, \dots, z_{r+j}\}$$

where  $\langle x_k | x_\ell \rangle_s = 0 = \langle z_k | z_\ell \rangle_s$  and  $\langle x_k | z_\ell \rangle_s = \delta_{k,\ell}$ . In particular,  $D = C \cap C^{\perp_s} = \langle z_{r+1}, \dots, z_{r+s} \rangle$ . It is possible to choose  $B$  such that it contains a vector  $z_k$  of minimum weight  $\text{swt}(C)$ .

*Proof:* Choose a basis  $\{z_1, \dots, z_{r+j}\}$  of a maximal isotropic subspace  $C_0$  of  $C$ . If  $C_0 \neq C$ , then we can choose a codeword  $x_1$  in  $C$  that is orthogonal to all of the  $z_k$  except one, say  $z_1$  (renumbering if necessary). By multiplying with a scalar in  $\mathbb{F}_p^\times$ , we may assume that  $\langle z_1 | x_1 \rangle_s = 1$ . If  $\langle C_0, x_1 \rangle \neq C$ , then one can repeat the process a finite number of times by choosing an  $x_k$  that is orthogonal to  $\{x_1, \dots, x_{k-1}\}$  until a basis of the desired form is found. ■

A subset  $\{z_k, x_k\}$  of  $C$  with  $\langle z_k | x_k \rangle_s = 1$  is called a hyperbolic pair. Thus, in the proof of the previous lemma, one chooses in each step a hyperbolic pair that is orthogonal to the previously chosen hyperbolic pairs.

*Theorem 16 ('Rain on your Parade Theorem'):* Let  $\mathbb{F}_q$  be a finite field of characteristic  $p$ . An  $((n, K, R > 1, d))_q$  Clifford subsystem code  $Q$  implies the existence of an  $((n, K, R/p, d))_q$  Clifford subsystem code  $Q_s$ . If  $Q$  is exactly pure to  $d'$ , then the subsystem code  $Q_s$  can be chosen such that it is exactly pure to  $d'$  as well.

*Proof:* By Theorem 1, there exists an additive code  $C \leq \mathbb{F}_q^{2n}$  with subcode  $D = C \cap C^{\perp_s}$  such that  $K = q^n / (|D||C|)^{1/2}$ ,  $R = (|C|/|D|)^{1/2}$ ,  $d = \text{swt}(D^{\perp_s} \setminus C)$ , and  $d' = \text{swt}(C)$ . By Lemma 15, one can find a  $\mathbb{F}_p$ -basis  $B$  of the form  $B = \{z_1, x_1; \dots; z_r, x_r; z_{r+1}, \dots, z_{r+j}\}$  such that  $\langle x_k | x_\ell \rangle_s = 0 = \langle z_k | z_\ell \rangle_s$  and  $\langle x_k | z_\ell \rangle_s = \delta_{k,\ell}$ . Notice that  $D = C \cap C^{\perp_s} = \langle z_{r+1}, \dots, z_{r+j} \rangle$  by Lemma 15.

Let  $C_s$  be the additive subcode of  $C$  given by  $C_s = \text{span}_{\mathbb{F}_p}(B \setminus \{x_r\})$ . Then  $D_s = C_s \cap C_s^{\perp_s} = \langle z_r, \dots, z_{r+j} \rangle$ . It follows that  $|C_s| = |C|/p$  and  $|D_s| = p|D|$ . Therefore,  $C_s$  defines a subsystem code  $Q_s = A_s \otimes B_s$  such that  $\dim A_s = q^n / (|C_s||D_s|)^{1/2} = K$  and  $\dim B_s = (|C_s|/|D_s|)^{1/2} = R/p$ .

Since  $D_s^{\perp_s} \subset D^{\perp_s}$ , any minimum weight codeword  $c \in D_s^{\perp_s} \setminus C_s$  must be either in  $D^{\perp_s} \setminus C$  or  $C$ . If it is in  $D^{\perp_s} \setminus C$ , then  $\text{swt}(c) \geq d$ . If it is in  $C$ , then it is a linear combination of elements in  $B \setminus \{x_r\}$ , since  $x_r \notin D_s^{\perp_s}$ . This implies that  $c$  is contained in  $C_s$ , contradicting our assumption that  $c$  is in  $D_s^{\perp_s} \setminus C_s$ . Therefore,  $\text{swt}(D_s^{\perp_s} \setminus C_s) \geq d$  and we can conclude that  $Q_s$  has minimum distance  $\geq d$ .

For the purity statement, recall that  $D \subset D_s \subseteq C_s \subset C$ . The subsystem code  $Q$  is exactly pure to  $d' = \text{swt}(C)$ . If  $\text{swt}(D) = d'$ , then  $\text{swt}(C_s) = d'$ ; otherwise,  $\text{swt}(C \setminus D) = d'$  and we can choose  $z_{r+1}$  such that  $\text{swt}(z_{r+1}) = d'$ . Then the subsystem code  $Q_s$  is exactly pure to  $\text{swt}(C_s) = d'$ . ■

*Corollary 17:* An  $((n, K, R, d))_q$  Clifford subsystem code that is exactly pure to  $d'$  implies the existence of an  $((n, K, \geq d))_q$  stabilizer code that is (exactly) pure to  $d'$ .

*Proof:* The corollary follows by repeatedly applying Theorem 16 to the  $((n, K, R, d))_q$  code and the derived code

until the dimension of the gauge subsystem is reduced to one. ■

We know that the MDS stabilizer codes arise from classical MDS codes. In fact, the stabilizer code is MDS if and only if the associated classical code is MDS. We can therefore hope that good subsystem codes can be obtained from classical MDS codes. We show that the resulting subsystem codes must be pure.

*Lemma 18:* If an  $((n, K > 1, R > 1, d))_q$  subsystem code is constructed from an MDS code, then the resulting code is pure.

*Proof:* Assume that  $C \subseteq \mathbb{F}_{q^2}^n$  is an  $[n, k, n - k + 1]_{q^2}$  code. If  $C^{\perp_a} \subseteq C$ , then  $K = 1$  contrary to our assumption. So assume that  $C^{\perp_a} \not\subseteq C$ . Let  $k > n - k$ . Then  $D = C \cap C^{\perp_a}$  must be smaller than  $C^{\perp_a}$ . And  $\dim D \leq n - k - 1$ . Hence  $\text{wt}(D^{\perp_a}) \leq (n - k - 1) + 1 = n - k < n - k + 1 = \text{wt}(C)$ . Hence the subsystem code is pure. Now assume that  $k \leq n - k$ . Now it is possible that  $C \subseteq C^{\perp_a}$ . If  $C \subseteq C^{\perp_a}$ , then  $R = 1$ . So  $C \not\subseteq C^{\perp_a}$ . Now  $\dim D \leq k - 1$  from which it follows that  $\text{wt}(D^{\perp_a}) \leq k \leq n - k < n - k + 1 = \text{wt}(C)$ . It follows that the subsystem code is pure. ■

## VI. STABILIZER VERSUS SUBSYSTEM CODES

In this section, we make a rigorous comparison between stabilizer codes and subsystem codes. Strictly speaking, subsystem codes contain the class of stabilizer codes; thus, in this section, we assume that the subsystem codes have a co-subsystem of dimension greater than 1.

Clearly, there are difficulties in comparing the two classes of codes. Our “rain on your parade” theorem shows that Clifford subsystem codes cannot have higher distances than stabilizer codes. Their main edge lies in simpler error recovery schemes. We can quantify this in terms of the number of syndrome measurements required for error-correction. This is not necessarily the best method to compare the decoding complexity. However, it is certainly a reasonable measure if both codes use table lookup decoding. In the absence of any special algorithms for subsystem codes, we will proceed with this as the metric for comparison.

### A. Improving Upon Quantum MDS Codes

In this subsection, we want to settle whether or not there exist subsystem code with parameters  $[[n, n - 2d + 2, r > 0, d]]_q$ . It turns out that the bounds that we have derived in Section IV will help in answering this question. Our best bounds are restricted to pure codes. Fortunately, it turns out that all subsystem codes with parameters  $((n, q^{n-2d+2}, R, d))_q$  are pure.

*Theorem 19:* Any  $((n, q^{n-2d+2}, R, d))_q$  Clifford subsystem code is pure.

*Proof:* If  $R = 1$ , then the claim follows from the fact that quantum MDS codes are pure, see [12].

Seeking a contradiction, we assume that there exists an impure subsystem code with parameters  $((n, q^{n-2d+2}, R, d))_q$ , exactly pure to  $d' < d$  and  $R > 1$ . It follows from Corollary 17 that it is possible to construct a stabilizer code with distance  $\geq d$  that is (exactly) pure to  $d'$ . Then the resulting

stabilizer code has the parameters  $((n, q^{n-2d+2}, d))_q$  and is impure. But we know that all quantum MDS codes are pure [12], see also [5, Corollary 60]. This implies that  $d' \geq d$  contradicting the fact that  $d' < d$ ; hence, every  $((n, q^{n-2d+2}, R, d))_q$  code is pure. ■

The next theorem explains why Poulin did not have any luck in finding an  $[[5, 1, r > 0, 3]]_2$  subsystem code.

**Theorem 20:** There do not exist any Clifford subsystem codes with parameters  $((n, q^{n-2d+2}, R > 1, d))_q$ . In particular, there do not exist any  $[[n, n-2d+2, r > 0, d]]_q$  Clifford subsystem codes.

*Proof:* Seeking a contradiction, we assume that a subsystem code with parameters  $((n, q^{n-2d+2}, R > 1, d))_q$  exists. By Theorem 19, an  $((n, q^{n-2d+2}, R, d))_q$  subsystem code must be pure. It follows from Theorem 11 that a pure subsystem code with these parameters must satisfy

$$q^{n-2d+2}R \leq q^{n-2d+2}.$$

Therefore, we must have  $R = 1$ , contradicting our assumption  $R > 1$ . ■

### B. Better Than Quantum MDS Codes

In this subsection, we compare once again quantum MDS stabilizer codes against subsystem codes. We require that both codes are able to encode the same amount of information and have the same distance. However, this time, we do not restrict the length of the codes. Our goal is to determine whether the subsystem code can improve upon an optimal quantum MDS stabilizer code by fewer syndrome measurements.

We insist that the codes are  $\mathbb{F}_q$ -linear, since in this case the number of syndrome measurements can be directly obtained from the code parameters. Indeed, recall that an  $\mathbb{F}_q$ -linear  $[[n, k, r, d]]_q$  subsystem code requires  $n - k - r$  syndrome measurements, and an  $\mathbb{F}_q$ -linear  $[[n', k', d']]_q$  stabilizer code requires  $n' - k'$  syndrome measurements.

**Theorem 21:** If there exists an  $\mathbb{F}_q$ -linear  $[[k+2d-2, k, d]]_q$  quantum MDS stabilizer code, then an  $\mathbb{F}_q$ -linear  $[[n, k, r, d]]_q$  subsystem code satisfying

$$k + r \leq n - 2d + 2 \quad (1)$$

cannot require fewer syndrome measurements than the stabilizer code.

[We remark that any pure  $[[n, k, r, d]]_q$  subsystem code satisfies the inequality (1) by Theorem 11.]

*Proof:* Seeking a contradiction, we assume that the subsystem code requires fewer syndrome measurements than the quantum MDS code, that is, we assume that  $k + 2d - 2 - k > n - k - r$ . This implies that  $k + r > n - 2d + 2$ , contradicting our assumption that  $k + r \leq n - 2d + 2$ . ■

Now, we can partially answer the question when an  $\mathbb{F}_q$ -linear  $[[n, k, r, d]]_q$  subsystem code will lead to better error recovery schemes than the quantum MDS codes.

**Corollary 22:** Suppose that an  $\mathbb{F}_q$ -linear  $[[k + 2d - 2, k, d]]_q$  quantum MDS code  $Q$  exists. Then an  $\mathbb{F}_q$ -linear  $[[n, k, r, d]]_q$  subsystem code that beats the the stabilizer code  $Q$  must be impure and must satisfy  $k + r > n - 2d + 2$ .

*Proof:* We know from Theorem 11 that all pure  $[[n, k, r, d]]_q$  codes satisfy  $k + r \leq n - 2d + 2$ . But Theorem 21 implies that such a code cannot have fewer syndrome measurements than the  $\mathbb{F}_q$ -linear MDS code. Hence, the subsystem code, if it is better, must be impure and have  $k + r > n - 2d + 2$ . ■

### C. Better Than Optimal non-MDS Stabilizer Codes

We know that MDS codes do not exist for all lengths, so it is reasonable to consider optimal stabilizer codes that are non-MDS. In this case, the comparison is slightly more complicated. An  $[[n, k, r, d]]_q$  subsystem code could be better than an optimal  $[[n', k, d]]_q$  stabilizer code. That in itself does not guarantee that the class of subsystem codes is superior to the class of stabilizer codes.

For instance, the shortest code to encode 2 qubits with distance 3 is  $[[8, 2, 3]]_2$  (see [3]). Suppose that an  $[[8, 2, 1, 3]]_2$  code exists. This subsystem code requires only  $8 - 2 - 1 = 5$  syndrome measurements as against the  $8 - 2 = 6$  measurements of the optimal stabilizer code. To conclude that the subsystem codes are better than stabilizer codes would be premature, for there exists an  $[[8, 3, 3]]_2$  code (cf. [3]) that requires  $8 - 3 = 5$  syndrome measurements and encodes one more qubit than the subsystem code. It is therefore necessary to compare the subsystem code with all optimal  $[[n', k' \geq k, d]]_q$  stabilizer codes, where  $n'$  ranges from  $n - r$  to  $n$ . Only if the subsystem code requires fewer syndrome measurements in each case, then we can conclude that the class of subsystem codes leads to better error recovery schemes.

We do not know precisely the properties of such subsystem codes. For instance, we do not know if such subsystem code is required to be impure or if it must satisfy  $k + r > n - 2d + 2$ .

Next, we turn our attention to a slightly different question which shows that in general whenever good subsystem codes exist, good stabilizer codes also exist.

### D. Subsystem Codes and Stabilizer Codes of Comparable Performance

The reader will perhaps wonder why one cannot simply discard the gauge subsystem to obtain a shorter quantum code without sacrificing distance or dimension. The reason why we cannot do so is because, in general, there is no one to one correspondence between the gauge qudits and the physical qudits. Yet, our intuition is not entirely misguided as the following result will show.

**Theorem 23:** Let  $\mathbb{F}_q$  be finite field of characteristic  $p$  and  $1 < q^k \leq q^n$  a power of  $p$ . Let  $r$  be an integer such that  $0 < r < n$ , and

$$\sum_{j=1}^{d-1} \binom{n}{j} (q^2 - 1)^j (q^{n+k+r} - q^{n+r-k}) < (p-1)(q^{2n} - 1)$$

holds, then there exist both an  $((n, q^k, q^r, \geq d))_q$  Clifford subsystem code and an  $((n - r, q^k, \geq d))_q$  stabilizer code.

*Proof:* By hypothesis

$$\sum_{j=1}^{d-1} \binom{n}{j} (q^2 - 1)^j < (p-1) \frac{(q^{2n} - 1)}{(q^{n+k+r} - q^{n+r-k})}$$

holds and Theorem 7 implies the existence of an  $((n, q^k, q^r, \geq d))_q$  Clifford subsystem code. We can rewrite the RHS of the inequality as

$$\begin{aligned} \text{RHS} &= (p-1) \frac{q^{n-r} - q^{-n-r}}{q^k - q^{-k}}, \\ &= (p-1) \frac{q^{n-r} - q^{-n+r}}{q^k - q^{-k}} + (p-1) \frac{q^{-n+r} - q^{-n-r}}{q^k - q^{-k}}, \\ &= (p-1) \frac{q^{n-r} - q^{-n+r}}{q^k - q^{-k}} + \underbrace{(p-1) \frac{q^{-n+r} - q^{-n-r}}{q^k - q^{-k}}}_{\leq 1, \text{ if } r > 0}. \end{aligned}$$

Now under the assumption  $r < n$ , we obtain a lower bound for LHS as follows.

$$\begin{aligned} \binom{n}{1} (q^2 - 1) + \sum_{j=2}^{d-1} \binom{n-r}{j} (q^2 - 1)^j \\ = \sum_{j=1}^{d-1} \binom{n-r}{j} (q^2 - 1)^j + r(q^2 - 1), \\ \leq \sum_{j=1}^{d-1} \binom{n}{j} (q^2 - 1)^j = \text{LHS}. \end{aligned}$$

Since we know that  $\text{LHS} < \text{RHS}$  we can write

$$\begin{aligned} \sum_{j=1}^{d-1} \binom{n-r}{j} (q^2 - 1)^j + \underbrace{r(q^2 - 1)}_{> 1, \text{ if } r > 0} \\ < (p-1) \frac{q^{n-r} - q^{-n+r}}{q^k - q^{-k}} + \underbrace{(p-1) \frac{q^{-n+r} - q^{-n-r}}{q^k - q^{-k}}}_{\leq 1, \text{ if } r > 0}, \\ \sum_{j=1}^{d-1} \binom{n-r}{j} (q^2 - 1)^j < (p-1) \frac{q^{n-r} - q^{-n+r}}{q^k - q^{-k}}. \end{aligned}$$

Then by Corollary 8, there exists an  $((n-r, q^k, \geq d))_q$  stabilizer code. ■

While they might differ in their distance, the preceding theorem indicates that in many cases, whenever a good subsystem code exists, then there will also exist a good stabilizer code encoding as much information and having comparable distance and of shorter length. The assumption of integral  $r$  may not be much of a restriction in light of Theorem 16.

In comparing the complexity of the error recovery schemes for the two codes, we run into a small problem since we do not know if the codes are  $\mathbb{F}_q$ -linear. Actually, if we use the stronger result of [5, Lemma 31] and insist that  $n \equiv k \pmod{2}$ , then we can show that the stabilizer code is  $\mathbb{F}_q$ -linear. This guarantees that the stabilizer code will require  $n-k-r$  syndrome measurements which is comparable to that of an  $\mathbb{F}_q$ -linear subsystem code. It appears then, quite often, subsystem codes do not offer any gains in error recovery, as there will exist a corresponding stabilizer code that encodes as many qudits, of similar distance and equal complexity of decoding.

## VII. CONCLUSION

In this paper we investigated subsystem codes and their connections to classical codes. We derived lower and upper bounds on the parameters of the subsystem codes. We settled the question whether or not there exist  $[[n, n-2d+2, r > 0, d]]_q$  subsystem codes exist. We showed that pure  $\mathbb{F}_q$ -linear subsystem codes do not lead to any reduction in complexity of error recovery as compared with an  $\mathbb{F}_q$ -linear MDS stabilizer code of equal capability. As a consequence we concluded that the subsystem codes that outperform the quantum MDS codes must be impure. Further, we showed that quite often the existence of a subsystem code implies the existence of a stabilizer code of comparable performance and complexity of error recovery.

## REFERENCES

- [1] D. Bacon. Operator quantum error correcting subsystems for self-correcting quantum memories. *Phys. Rev. A*, 73(012340), 2006.
- [2] W. Bosma, J.J. Cannon, and C. Playoust. The Magma algebra system I: The user language. *J. Symb. Comp.*, 24:235–266, 1997.
- [3] A.R. Calderbank, E.M. Rains, P.W. Shor, and N.J.A. Sloane. Quantum error correction via codes over  $\text{GF}(4)$ . *IEEE Trans. Inform. Theory*, 44:1369–1387, 1998.
- [4] L.C. Grove. *Classical Groups and Geometric Algebra*. Graduate Studies in Mathematics. American Mathematical Society, 2001.
- [5] A. Ketkar, A. Klappenecker, S. Kumar, and P.K. Sarvepalli. Nonbinary stabilizer codes over finite fields. To appear in *IEEE Trans. Inform. Theory*, November, 2006.
- [6] A. Klappenecker and P. K. Sarvepalli. Clifford code constructions of operator quantum error-correcting codes. 2006.
- [7] E. Knill. On protected realizations of quantum information. Eprint: quant-ph/0603252, 2006.
- [8] D. W. Kribs. A brief introduction to operator quantum error correction. Eprint: math/0506491, 2005.
- [9] D. W. Kribs, R. Laflamme, and D. Poulin. Unified and generalized approach to quantum error correction. *Phys. Rev. Lett.*, 94(180501), 2005.
- [10] D. W. Kribs, R. Laflamme, D. Poulin, and M. Lesosky. Operator quantum error correction. Eprint: quant-ph/0504189, 2005.
- [11] D. Poulin. Stabilizer formalism for operator quantum error correction. *Phys. Rev. Lett.*, 95(230504), 2005.
- [12] E.M. Rains. Nonbinary quantum codes. *IEEE Trans. Inform. Theory*, 45:1827–1832, 1999.